AN AFFINE SLICING APPROACH TO CERTAIN PAUCITY PROBLEMS

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Dedicated to Professor Heini Halberstam on the occasion of his retirement

1. Introduction

Estimates for the number of integral solutions of symmetric diagonal equations are of importance in numerous applications of the Hardy-Littlewood method (see Vaughan [15]). When the number of variables in such a system is not too large compared with the underlying degrees (in a sense we make precise below), it is conjectured that the number of diagonal solutions exceeds the corresponding number of non-diagonal solutions. In this paper we make a modest contribution towards the resolution of this widely held conjecture by obtaining an asymptotic formula for the number of solutions of a pair of equations, one cubic and one quadratic. Let $S(P)$ denote the number of solutions of the simultaneous diophantine equations

$$\begin{aligned} x_1^3+x_2^3+x_3^3&=y_1^3+y_2^3+y_3^3,\\ x_1^2+x_2^2+x_3^2&=y_1^2+y_2^2+y_3^2, \end{aligned} \tag{1.1}$$

with $1\le x_i,y_i\le P$ $(1\le i\le 3)$, and let $T(P)$ denote the corresponding number of solutions with (x_1,x_2,x_3) a permutation of (y_1,y_2,y_3). In §2 below we establish the upper and lower bounds for $S(P)-T(P)$ contained in the following theorem.

Theorem 1. *Suppose that P is a positive real number. Then*

$$P\ll S(P)-T(P)\ll_\varepsilon P^{7/3+\varepsilon}. \tag{1.2}$$

In particular, one has the asymptotic formula

$$S(P)=6P^3+O_\varepsilon(P^{7/3+\varepsilon}).$$

We note that the upper bound $S(P)\ll_\varepsilon P^{3+\varepsilon}$ was established in [18, Theorem 4.1]. By using Hua [11, Theorem 3], it is shown in [19, Theorem 1] that the P^ε occurring in the latter bound may be replaced by a power of $\log P$, and indeed such

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an improvement may also be effected in (1.2). We note also that the argument of §2, with only trivial modification, may be applied to obtain the conclusion (1.2) with the definitions of $S(P)$ and $T(P)$ modified to count solutions with $|x_i| \le P$ and $|y_i| \le P$ $(1 \le i \le 3)$.

Theorem 1 establishes a special case of the conjecture alluded to above, which we now state formally. Let $k_1, \dots, k_t$ be integers with $1 \le k_1 < \dots < k_t$, and let s be a positive integer. Denote by $N_s(P;\mathbf{k})$ the number of integral solutions of the system of equations

$$\sum_{i=1}^{s}(x_i^{k_j} - y_i^{k_j}) = 0 \quad (1 \le j \le t), \tag{1.3}$$

with $1 \le x_i, y_i \le P$ $(1 \le i \le s)$, and let $T_s(P)$ denote the corresponding number of *diagonal solutions*, which is to say the number of solutions of (1.3) counted by $N_s(P;\mathbf{k})$ in which the x_i are a permutation of the y_j. Then it is conjectured that when $s < k_1 + \dots + k_t$, one has

$$N_s(P;\mathbf{k}) - T_s(P) = o_{s,\mathbf{k}}(T_s(P)), \tag{1.4}$$

and hence that there is a paucity of non-diagonal solutions of the diagonal system (1.3). In the current state of knowledge there seems to be no hope of establishing the conjectured estimate (1.4) in any cases with $t+1 < s < k_1 + \dots + k_t$. However, the estimate (1.4) follows in all cases with $1 \le s \le t$ from a theorem of Steinig [14], which shows that

$$N_s(P;\mathbf{k}) = T_s(P) \quad (1 \le s \le t).$$

Thus the situation of greatest current interest is that in which $s = t+1$ and $k_t > 2$. There has been considerable progress for certain subcases of this case of the conjecture. In particular, Hooley [7, 8, 9, 10] and Greaves [3, 4] have used sieve methods, combined with estimates of Weil and Deligne, to establish (1.4) when $t = 1$, $s = 2$ and $k_t > 2$. For the special subcases in which $s = t+1$ and $\mathbf{k} = (1, 2, \dots, t)$ or $\mathbf{k} = (1, 2, \dots, t-1, t+1)$, Vaughan and Wooley [17] have established (1.4), with rather strong bounds for the difference $N_s(P;\mathbf{k}) - T_s(P)$, through an essentially elementary argument. In the simplest remaining subcases where $s = 3$, $t = 2$, $\mathbf{k} = (1, k)$ and $k > 3$, Greaves [5] has very recently established (1.4) by developing further the sieve methods of Hooley and himself. We note also that Skinner and Wooley [12, 13] have developed a homogeneous slicing argument which, when combined with estimates of Bombieri and Pila [1], yields more precise estimates than those of Hooley [10] and Greaves [4, 5] for larger k_t.

In order to prove the upper bound of (1.2), which establishes the above conjecture in a hitherto unresolved case, we use an affine slicing argument closely related to that used in our previous work on sums of two cubes [20]. For a fixed integer h we consider the number, $U(P,h)$, of solutions of the system (1.1) counted by $S(P)$ which satisfy the additional condition

$$x_1 + x_2 + x_3 - y_1 - y_2 - y_3 = h.$$

For solutions of this new system it is almost immediate that the x_i are a permutation of the y_j if and only if h is zero. Thus we are able to estimate $S(P) - T(P)$ by bounding $U(P,h)$ for non-zero h, and then summing over all possible non-zero values of h. By comparing the new system of equations with the system

$$\sum_{i=1}^{4}(x_i^j - y_i^j) = 0 \quad (1 \le j \le 3),$$

we are led naturally to identities of a multiplicative nature (see, for example, [17, §10]). Hence, by extracting common factors between variables, we are finally able to bound $U(P,h)$ in terms of the number of solutions of a binary quadratic equation in a manner analogous to that applied in the argument of [20].

The lower bound in Theorem 1 is almost immediate from the observation that

$$\begin{aligned} 66^3 + 29^3 + 26^3 &= 62^3 + 45^3 + 2^3, \\ 66^2 + 29^2 + 26^2 &= 62^2 + 45^2 + 2^2, \end{aligned}$$

since the family of solutions $(\mathbf{x}, \mathbf{y}) = (66t, 29t, 26t, 62t, 45t, 2t)$, and permutations thereof, contribute at least $6[P/66] \gg P$ solutions to $S(P) - T(P)$.

In §3 below we review the affine slicing argument used in [20] to establish Hooley's bound on the number of non-trivial solutions of the equation $x_1^3 + x_2^3 = y_1^3 + y_2^3$. It is evident that the argument of §2 is a natural extension of that sketched in §3, and thus it is tempting to speculate concerning possible extensions of such methods to larger systems of equations. It seems possible that our affine slicing argument might be applied to resolve the cases of the conjecture described above in which $s = t+1$ and $\mathbf{k} = (2, 3, \dots, t+1)$ or $\mathbf{k} = (2, \dots, t, t+2)$. For in these cases, an affine slice generates a system similar to those considered in §§8 and 9 of Vaughan and Wooley [17]. Unfortunately the equations deriving from the subsequent analysis will no longer be so simple as the binary quadratics resulting from the analysis of this paper, and that contained in [20]. One would presumably be obliged to make use of the estimates of Bombieri and Pila [1], and this entails the difficult task of developing certain criteria for absolute irreducibility. We finish §3 by noting two conditional improvements on Hooley's bound which depend on certain conjectured estimates for the number of rational or integral points on elliptic curves.

Throughout this paper ε will denote a sufficiently small positive number, and $\ll$ and $\gg$ will denote Vinogradov's well-known notation.

2. The proof of Theorem 1

Let $U_1(P,h)$ denote the number of solutions of the simultaneous equations

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 &= y_1^3 + y_2^3 + y_3^3, \\ x_1^2 + x_2^2 + x_3^2 &= y_1^2 + y_2^2 + y_3^2, \\ x_1 + x_2 + x_3 &= y_1 + y_2 + y_3 + h, \end{aligned} \tag{2.1}$$

with $1 \le x_i, y_i \le P$ $(1 \le i \le 3)$. When $h = 0$, it follows from Newton's formulae on the roots of polynomials that any solution $\mathbf{x}$, $\mathbf{y}$ of (2.1) satisfies the condition that (x_1, x_2, x_3) is a permutation of (y_1, y_2, y_3). Thus

$$S(P) - T(P) = \sum_{1 \le |h| \le 3P} U_1(P, h) \ll (\log 2P) \max_{1 \le H \le P} \sum_{H \le |h| \le 3H} U_1(P, h). \tag{2.2}$$

Let h be non-zero, and let $\mathbf{x}$, $\mathbf{y}$ be a solution of the system (2.1) counted by $U_1(P, h)$. Notice that by relabelling variables we may suppose that $y_3 = \min_{1 \le i \le 3}\{x_i, y_i\}$. We note also that if $x_i = y_j$ for any i and j, then by relabelling variables one may suppose that $i = j = 3$, and so (2.1) implies that $x_1^k + x_2^k = y_1^k + y_2^k$ $(k = 2, 3)$. Thus, either by a straightforward manipulation, or by reference to Steinig [14], one deduces that (x_1, x_2, x_3) is a permutation of (y_1, y_2, y_3), and from (2.1) we obtain $h = 0$. This provides a contradiction which shows that when $h \ne 0$, one has $x_i = y_j$ for no i and j.

Define the polynomials s_j for $1 \le j \le 3$ by

$$s_j(x_1, x_2, x_3, y_3) = x_1^j + x_2^j + x_3^j - y_3^j.$$

Then it follows from (2.1) that

$$\begin{aligned} &s_1(\mathbf{x}, y_3)^3 - 3s_1(\mathbf{x}, y_3)s_2(\mathbf{x}, y_3) + 2s_3(\mathbf{x}, y_3) \\ &\qquad = (y_1 + y_2 + h)^3 - 3(y_1 + y_2 + h)(y_1^2 + y_2^2) + 2(y_1^3 + y_2^3). \end{aligned} \tag{2.3}$$

On making the substitution

$$u_i = x_i - y_3 \quad (1 \le i \le 3), \tag{2.4}$$

one readily deduces that the x_i are uniquely determined from the u_j together with y_3. Let $U_2(P, h)$ denote the number of solutions of the system

$$12u_1u_2u_3 + h^3 = 3h(2y_1 + h)(2y_2 + h), \tag{2.5}$$

$$(u_1 + y_3)^2 + (u_2 + y_3)^2 + (u_3 + y_3)^2 = y_1^2 + y_2^2 + y_3^2, \tag{2.6}$$

$$u_1 + u_2 + u_3 + 2y_3 = y_1 + y_2 + h, \tag{2.7}$$

with

$$1 \le u_i, y_i \le P \quad (1 \le i \le 3), \tag{2.8}$$

and

$$h^4 = 12u_iu_j(h - u_i)(h - u_j) \tag{2.9}$$

for some i and j with $1 \le i < j \le 3$. Let $U_3(P, h)$ denote the corresponding number of solutions with the condition (2.9) replaced by

$$h^4 \ne 12u_iu_j(h - u_i)(h - u_j) \quad (1 \le i < j \le 3). \tag{2.10}$$

Then we deduce from (2.1), (2.3) and (2.4) that

$$U_1(P,h) \ll U_2(P,h) + U_3(P,h). \tag{2.11}$$

We divide into cases.

(i) We first estimate $U_2(P,h)$, noting that

$$U_2(P,h) = V_1(P,h) + V_2(P,h), \tag{2.12}$$

where $V_1(P,h)$ denotes the number of solutions counted by $U_2(P,h)$ in which

$$h^3 + 12u_1u_2u_3 = 0, \tag{2.13}$$

and $V_2(P,h)$ denotes the corresponding number of solutions with the condition (2.13) replaced by $h^3 + 12u_1u_2u_3 \neq 0$.

Let h be non-zero, and consider the number of solutions $\mathbf{u}$, $\mathbf{y}$ of the system (2.5)-(2.7) counted by $V_1(P,h)$. Since each of the u_i are divisors of h^3, by using standard estimates for the divisor function we find that the number of possible choices for u_1, u_2 and u_3 satisfying (2.13) is $O_\varepsilon(|h|^\varepsilon)$. Fix any one such choice, and fix any one of the $[P]$ possible choices for y_3. On writing

$$N = (u_1 + y_3)^2 + (u_2 + y_3)^2 + (u_3 + y_3)^2 - y_3^2,$$

we find from (2.6) that $y_1^2 + y_2^2 = N$. Then the number of possible choices for y_1 and y_2 is $O_\varepsilon(N^\varepsilon)$, whence

$$V_1(P,h) \ll_\varepsilon |h|^\varepsilon P^{1+\varepsilon}. \tag{2.14}$$

Next we consider, for fixed non-zero h, the number of solutions of (2.5)-(2.7) counted by $V_2(P,h)$. By rearranging variables we may suppose that (2.9) holds with $i = 1$ and $j = 2$. Thus, on using standard estimates for the divisor function, it follows from (2.9) that the number of possible choices for u_1 and u_2 is $O_\varepsilon(|h|^\varepsilon)$. Fix any one such choice, and fix any one of the $[P]$ possible choices for u_3. Then in view of (2.5), each of the numbers $2y_1 + h$ and $2y_2 + h$ is a divisor of the integer $h^3 + 12u_1u_2u_3$. Moreover for any solution counted by $V_2(P,h)$, the latter expression is non-zero. Consequently, again by using standard estimates for the divisor function, one has at most $O_\varepsilon((|h|P)^\varepsilon)$ possible choices for y_1 and y_2. Fix any one such. Then y_3 is uniquely determined by equation (2.7). On combining these estimates we deduce that $V_2(P,h) \ll_\varepsilon |h|^\varepsilon P^{1+\varepsilon}$, whence by (2.12) and (2.14), when h is non-zero one has

$$U_2(P,h) \ll_\varepsilon |h|^\varepsilon P^{1+\varepsilon}. \tag{2.15}$$

(ii) Next we estimate $U_3(P,h)$. Suppose that h is non-zero, and consider any solution $\mathbf{u}$, $\mathbf{y}$ of (2.5)-(2.7) counted by $U_3(P,h)$. On noting that

$$h^2 + 3h(y_1 + y_2) + 6y_1y_2 = h^2 - 3h(y_1 + y_2 + h) + 3\left((y_1 + y_2 + h)^2 - (y_1^2 + y_2^2)\right),$$

we deduce from (2.5) that

$$12u_1u_2u_3 = 2h\left(h^2 - 3h(y_1+y_2+h) + 3\left((y_1+y_2+h)^2 - (y_1^2+y_2^2)\right)\right),$$

whence from (2.6) and (2.7),

$$\begin{aligned}12u_1u_2u_3 =& 2h\left(h^2 - 3h(u_1+u_2+u_3+2y_3) + 3(u_1+u_2+u_3+2y_3)^2\right.\\ &\left. -3\left((u_1+y_3)^2+(u_2+y_3)^2+(u_3+y_3)^2-y_3^2\right)\right).\end{aligned}$$

On making the substitution

$$t = 2y_3 + u_1 + u_2 + u_3 - h,$$

and writing

$$Q(u_1,u_2,u_3) = u_1^2 - 2u_1u_2 + u_2^2 - 2u_2u_3 + u_3^2 - 2u_3u_1,$$

we deduce that

$$12u_1u_2u_3 = h\left(3t^2 - 3Q(u_1,u_2,u_3) - h^2\right). \tag{2.16}$$

Moreover y_3 is uniquely determined from t, u_1, u_2, u_3 and h. Also, given y_3, u_1, u_2, u_3 and h, one may determine y_1 and y_2 uniquely from the equations (2.6) and (2.7), up to a permutation, by solving the implicit quadratic equation. Thus we deduce that

$$U_3(P,h) \ll W_1(P,h), \tag{2.17}$$

where $W_1(P,h)$ denotes the number of solutions $\mathbf{u}$, t of the equation (2.16) satisfying (2.10), with $\mathbf{u}$ satisfying (2.8) and

$$|t| \le 8P. \tag{2.18}$$

Let h be non-zero, and let $\mathbf{u}$, t be any solution of the equation (2.16) counted by $W_1(P,h)$. Notice that by relabelling variables we may suppose that the pairwise common factors of the u_i and h satisfy the conditions

$$(u_1,h) \ge \max\{(u_2,h),(u_3,h)\} \quad \text{and} \quad (u_2,h/(u_1,h)) \ge (u_3,h/(u_1,h)).$$

Write

$$d_1 = (u_1,h), \quad d_2 = (u_2,h/d_1), \quad d_3 = (u_3,h/(d_1d_2)),$$
$$g = h/(d_1d_2d_3), \quad f = 12/g, \quad \text{and} \quad v_i = u_i/d_i \quad (1 \le i \le 3).$$

From (2.16) we see that f and g are integers. Thus, on recalling the above observation, together with the condition (2.10),

$$h = gd_1d_2d_3, \quad fg = 12, \quad d_1 \ge d_2 \ge d_3, \tag{2.19}$$

$$g^4(d_1d_2)^2d_3^4 \ne 12v_1v_2(gd_2d_3 - v_1)(gd_1d_3 - v_2), \tag{2.20}$$

and $u_i = v_i d_i$ $(1 \le i \le 3)$. On substituting into (2.16), one deduces that

$$W_1(P,h) \ll W_2(P,h), \tag{2.21}$$

where $W_2(P,h)$ denotes the number of solutions of the equation

$$fv_1v_2v_3 = 3t^2 - 3Q(d_1v_1, d_2v_2, d_3v_3) - (gd_1d_2d_3)^2, \tag{2.22}$$

with $\mathbf{v}$, $\mathbf{d}$, f, g and t satisfying (2.18)-(2.20) and

$$1 \le v_i \le P/d_i \quad (1 \le i \le 3). \tag{2.23}$$

Let $W_3(P;\mathbf{d}, f, g, v_1, v_2)$ denote the number of solutions of the equation (2.22) with t and v_3 satisfying (2.18) and (2.23). Then it follows that

$$W_2(P,h) \ll \sum_{gf=12} \sum_{\substack{gd_1d_2d_3=h \\ d_3=\min_{1\le i\le 3} d_i}} \sum_{1\le v_1\le P/d_1} \sum_{1\le v_2\le P/d_2} W_3(P;\mathbf{d}, f, g, v_1, v_2), \tag{2.24}$$

where the last summation is restricted by the condition (2.20).

For fixed $\mathbf{d}$, f, g, v_1 and v_2 arising in the summation of (2.24), the equation (2.22) may be written in the form

$$6t^2 - 6d_3^2v_3^2 - 2Av_3 = B,$$

where

$$A = fv_1v_2 - 6d_3(d_1v_1 + d_2v_2) \quad \text{and} \quad B = 6(d_1v_1 - d_2v_2)^2 + 2(gd_1d_2d_3)^2.$$

On completing the square, we obtain $\xi^2 - \eta^2 = n$, where

$$\xi = 6d_3t, \quad \eta = 6d_3^2v_3 + A \quad \text{and} \quad n = 6d_3^2B - A^2. \tag{2.25}$$

Following a little computation, one finds that

$$n = \frac{12}{g^2}\left(g^4d_1^2d_2^2d_3^4 - 12v_1v_2(gd_2d_3 - v_1)(gd_1d_3 - v_2)\right).$$

Thus n is non-zero, since for values of v_1, v_2, g, d_1, d_2 and d_3 occurring in the summation of (2.24), one has the inequality (2.20). Further, the map $(t, v_3) \to (\xi, \eta)$ described by (2.25) is one-to-one. It follows that $W_3(P;\mathbf{d}, f, g, v_1, v_2)$ is bounded above by the number of solutions of the equation $\xi^2 - \eta^2 = n$. But if ξ, η is any solution of the latter equation, then $\xi - \eta$ and $\xi + \eta$ are both divisors of the non-zero integer n, whence

$$W_3(P;\mathbf{d}, f, g, v_1, v_2) \ll_\varepsilon P^\varepsilon. \tag{2.26}$$

On combining (2.24) and (2.26), we obtain

$$W_2(P,h) \ll_\varepsilon P^\varepsilon \sum_{gf=12} \sum_{\substack{gd_1d_2d_3=h \\ d_3=\min\{d_1,d_2,d_3\}}} P^2/(d_1d_2).$$

The summation condition on d_3 ensures that $d_1d_2 \gg |h|^{2/3}$, whence by using an elementary estimate for the divisor function we obtain

$$W_2(P,h) \ll_\varepsilon P^{2+2\varepsilon}|h|^{-2/3}.$$

Thus, on recalling (2.2), (2.11), (2.15), (2.17) and (2.21), we deduce that

$$S(P) - T(P) \ll_\varepsilon P^{3\varepsilon} \max_{1\le H\le P} H\left(P^2H^{-2/3} + PH^\varepsilon\right) \ll_\varepsilon P^{7/3+4\varepsilon}.$$

This completes the proof of Theorem 1.

3. Review of sums of two cubes

In [20] we used an affine slicing argument in work on sums of two cubes. Since the argument applied there is closely related to that used in §2 of the present paper, we briefly sketch below the former argument in order that the similarities be made more apparent. In Theorem 2 below we provide a conclusion slightly more general than that of [20, Theorem 2].

Theorem 2. *Let $N(P)$ denote the number of solutions of the equation*

$$x_1^3+x_2^3=x_3^3+x_4^3, \tag{3.1}$$

with $|x_i|\le P$ $(1\le i\le 4)$, and let $N^*(P)$ denote the corresponding number of solutions with $x_1\ne x_j$ $(j=3,4)$ and $x_1+x_2\ne 0$. Then

$$N^*(P)\ll_\varepsilon P^{5/3+\varepsilon},$$

and consequently one has the asymptotic formula

$$N(P)=12P^2+O_\varepsilon(P^{5/3+\varepsilon}).$$

The main conclusion of Theorem 2 was first obtained by Hooley [9] through the use of an ingenious sieve method which made use of estimates of Deligne. It is, however, possible to prove Theorem 2 using nothing more sophisticated than the classical theory of binary quadratic forms, as we sketch below.

We start by using an affine slicing argument, similar to that applied at the start of §2, to show that

$$N^*(P)\ll(\log 2P)\max_{1\le H\le P}\sum_{H\le|h|\le 4H}N^*(P,h),$$

where $N^*(P,h)$ denotes the number of solutions of the system

$$\begin{aligned}12u_1u_2u_3&=h(3y^2+h^2),\\ u_1+u_2+u_3&=y+h,\end{aligned}$$

with

$$1\le|u_i|\le 2P\quad(1\le i\le 3),\quad |y|\le 6P,\quad h\ne u_i\quad(1\le i\le 3).$$

On extracting pairwise common factors between the u_i and h, it follows as in the argument leading to (2.24) that

$$N^*(P)\ll_\varepsilon P^\varepsilon\max_{1\le H\le P}\sum_{gf=12}\sum_{H\le|h|\le 4H}\sum_{\substack{gd_1d_2d_3=h\\ d_3=\max_{1\le i\le 3}d_i}}\sum_{\substack{1\le|v_3|\le 2P/d_3\\ v_3\ne gd_1d_2}}M(P;\mathbf{d},f,g,v_3), \tag{3.2}$$

where $M(P;\mathbf{d},f,g,v_3)$ denotes the number of solutions of the equation

$$fv_1v_2v_3=3(d_1v_1+d_2v_2+d_3v_3-gd_1d_2d_3)^2+(gd_1d_2d_3)^2, \tag{3.3}$$

with $1\le |v_i|\le 2P/d_i$ $(i=1,2)$. On noting that the equation (3.3) provides a binary quadratic relation between v_1 and v_2, and that the summation condition $v_3\ne gd_1d_2$ ensures that the implicit quadratic polynomial is irreducible, one finds by using classical estimates (see, for example, Estermann [2]) that

$$M(P;\mathbf{d},f,g,v_3)\ll_\varepsilon P^\varepsilon.$$

Then from (3.2),

$$N^*(P)\ll_\varepsilon P^{2\varepsilon}\max_{1\le H\le P}\sum_{gf=12}\sum_{H\le |h|\le 4H}\sum_{\substack{gd_1d_2d_3=h\\ d_3=\max\{d_1,d_2,d_3\}}}P/d_3. \tag{3.4}$$

The summation condition on d_3 ensures that $d_3\gg H^{1/3}$, and so by using an elementary estimate for the divisor function we deduce that

$$N^*(P)\ll_\varepsilon P^{3\varepsilon}\max_{1\le H\le P}H(PH^{-1/3})\ll_\varepsilon P^{5/3+3\varepsilon}.$$

This completes our sketch of the proof of Theorem 2.

We note that two further ideas have been proposed which conditionally improve Hooley's estimate for $N^*(P)$. At a conference in Göttingen in September 1992, Heath-Brown [6] described an argument in which a homogeneous slice is applied to the equation (3.1), thus reducing it to the form $C(x_1,x_2,x_3)=0$, with $C(\mathbf{x})$ a homogeneous cubic polynomial. Moreover the slicing argument can be arranged so as to exclude the trivial solutions. When $C(\mathbf{x})=0$ is the equation of an elliptic curve, and the coefficients of C are bounded in absolute value by H, then it is conjectured that the number, $M_C(Q)$, of primitive integral solutions of the equation $C(\mathbf{x})=0$ with $|x_i|\le Q$ $(1\le i\le 3)$, satisfies

$$M_C(Q)\ll_\varepsilon (HQ)^\varepsilon. \tag{3.5}$$

Assuming the truth of this conjecture concerning the number of rational points on an elliptic curve, Heath-Brown's argument yields the bound

$$N^*(P)\ll_\varepsilon P^{4/3+\varepsilon}.$$

The conjectured estimate (3.5), if true, would appear to lie deep, and there seems at present to be little prospect of establishing such a powerful result.

A second idea stems from a more specialised slicing method of Skinner and Wooley [12], which bears some similarity to earlier sieving methods. So far as the problem at hand is concerned, their method requires a bound for the number, $M(Q;a,b,c,d,x)$, of integral solutions of the equation

$$a^2x(3y^2+d^2x^2)=z(3(bz-cx)^2+(adz)^2), \tag{3.6}$$

with $|z|\le P$ and $|y|\le P$ (see [12, equation (3.16)]). In the application, the numbers a,b,c,d,x are all non-zero integers bounded by a power of P. Rather than describe this method in detail, we curtail our discussions by noting that the argument of [12, Lemma 3.1] leading to [12, (3.16), (3.17) and (3.20)] shows that

$$N^*(P)\ll_\varepsilon P^{3/2+\varepsilon}\max_{a,b,c,d,x} M(2P;a,b,c,d,x). \tag{3.7}$$

When the curve described by (3.6) is singular, the group law is sufficiently simple that $M(2P;a,b,c,d,x)$ is easily estimated to be $O_\varepsilon(P^\varepsilon)$. Otherwise, the equation (3.6) is essentially the Weierstrass form of the equation of an elliptic curve. When the curve described by the equation

$$y^2=ax^3+bx^2+cx+d \tag{3.8}$$

is non-singular, it is conjectured that the number of integral points (x,y) on the curve with $|x|\le Q$ and $|y|\le Q$, is $O_\varepsilon((\max_{\alpha=a,b,c,d}|\alpha|Q)^\varepsilon)$. Subject to the truth of this conjecture, it follows from (3.7) that

$$N^*(P)\ll_\varepsilon P^{3/2+\varepsilon}.$$

This last conjecture on the number of integral points on an elliptic curve would appear to be rather less deep than the conjecture used by Heath-Brown concerning rational points. Indeed, in the case that the cubic polynomial in x in (3.8) has a rational root, even the classical theory of binary quadratic forms may be used to establish the desired conclusion, as we now demonstrate. For in such circumstances it suffices to consider the equation

$$y^2=x(ax^2+bx+c) \tag{3.9}$$

with $ac\ne 0$ and $b^2\ne 4ac$. Let (x,y) be any integral solution of (3.9) in which neither x nor y is zero. On writing $d=(y,x)$, $y_1=y/d$ and $x_1=x/d$, we obtain the new equation

$$dy_1^2=x_1(a(dx_1)^2+bdx_1+c).$$

Thus $x_1|dy_1^2$ with $(x_1,y_1)=1$. Consequently $x_1|d$, and on writing $d_1=d/x_1$, we obtain

$$d_1y_1^2=ad_1^2x_1^4+bd_1x_1^2+c. \tag{3.10}$$

It follows that the number of solutions of the equation (3.9) inside a box of size Q is bounded above by the number of solutions of the equation (3.10) with $1\le |d_1|\le Q$, $|x_1|\le (Q/d_1)^{1/2}$ and $|y_1|\le Q/(d_1x_1)$. But if (d_1,x_1,y_1) is any such solution, then

$d_1|c$, whence there are $O_\varepsilon(|c|^\varepsilon)$ possible choices for d_1. Fix any one such choice, and substitute $X = 2ad_1x_1^2 + b$. Then the number of possible choices for x_1 and y_1 is bounded above by the number of solutions (X, y_1) of the binary quadratic equation

$$X^2 - 4ad_1y_1^2 = b^2 - 4ac,$$

with $|X| \le 2|a|Q + |b|$ and $|y_1| \le Q$. Since $b^2 \ne 4ac$, the classical theory of binary quadratic forms shows that the latter number is $O_\varepsilon(|ac(|b|+1)|^\varepsilon \log(2Q))$ (see, for example, Estermann [2] or Vaughan and Wooley [16, Lemma 3.5]). Thus we have established the desired conclusion for the equation (3.9).

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